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Factorization method for difference equations of hypergeometric type on nonuniform lattices

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Abstract

We study the factorization of the hypergeometric-type difference equation of Nikiforov and Uvarov on nonuniform lattices. An explicit form of the raising and lowering operators is derived and some relevant examples are given.

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1. Introduction

In this paper we deal with the so-called factorization method (FM) of the hypergeometric-type difference equations on nonuniform lattices. The FM was used by Darboux [14] and Schrödinger [25, 26] to obtain the solutions of differential equations, and also by Infeld and Hull [16] for finding analytical solutions of certain classes of second-order differential equations. Later, Miller extended it to difference equations [20] and q-differences—in the Hahn sense— [21]. For more recent works see, for example, [4, 10, 11, 29, 30] and references therein.

The classical FM was based on the existence of so-called raising and lowering operators for the corresponding equation that allow one to find explicit solutions in a very easy way. Going further, Atakishiyev and coauthors [4, 6, 10] found the dynamical symmetry algebra related to the FM and differential or difference equations. Of special interest was the paper by Smirnov [27] in which the equivalence of the FM and the Nikiforov *et al* theory [23] was shown; furthermore, this paper pointed out that the aforementioned equivalence remains valid also for nonuniform lattices, as was shown in [28, 29]. In particular, in [29] a detailed study of the FM established its equivalence with the Nikiforov *et al* approach to difference equations [23]. Also, in [12], a special nonuniform lattice was considered: the author constructed the FM for the Askey–Wilson polynomials using the difference equation for the polynomials. In this paper we continue the research of the nonuniform lattice case. Following the idea of Bangerezako [12] for the Askey–Wilson polynomials and Lorente [18] for the classical continuous and discrete cases, we obtain the FM for general polynomial solutions of the hypergeometric difference equation on the general quadratic nonuniform lattice $x(s) = c_1q^s + c_2q^{-s} + c_3$. We use, as was suggested in [6, 27], not the polynomial solutions but the corresponding normalized functions, which is a more natural and useful approach. Thus, the method proposed here is the generalization of [12] and [18] to the aforementioned nonuniform lattice.

The structure of the paper is as follows. In section 2 we present some well known results on orthogonal polynomials on nonuniform lattices [7, 23, 24]. In section 3 we introduce the normalized functions and obtain some of their properties such as the lowering and raising operators that allow us, in section 4, to obtain the factorization for the second-order difference equation satisfied by such functions. Finally, in section 5, some relevant examples are worked out.

2. Some basic properties of the q-polynomials

Here, we summarize some of the properties of the q-polynomials that will prove useful later. For further information see, for example, [23].

We deal here with the *second-order difference equation of the hypergeometric type*

$$\begin{gathered}\sigma(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\left[\frac{\nabla y(s)}{\nabla x(s)}\right]+\tau(s)\frac{\Delta y(s)}{\Delta x(s)}+\lambda y(s)=0\\ \sigma(s)=\tilde\sigma(x(s))-\tfrac{1}{2}\tilde\tau(x(s))\Delta x(s-\tfrac{1}{2})\qquad \tau(s)=\tilde\tau(x(s))\end{gathered}\tag{1}$$

where $\nabla f(s) = f(s) - f(s-1)$ and $\Delta f(s) = f(s+1) - f(s)$ denote the backward and forward finite-difference derivatives, respectively, $\tilde\sigma(x(s))$ and $\tilde\tau(x(s))$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant. We use the following notation for the coefficients in the power expansions in $x(s)$ of $\tilde\sigma(s)$ and $\tilde\tau(s)$:

$$\tilde\sigma(s)\equiv\tilde\sigma[x(s)]=\frac{\tilde\sigma''}{2}x^2(s)+\tilde\sigma'(0)x(s)+\tilde\sigma(0)\qquad \tilde\tau(s)\equiv\tilde\tau[x(s)]=\tilde\tau'x(s)+\tilde\tau(0).\tag{2}$$

An important property of the above equation is that the *k-order difference derivative* of a solution $y(s)$ of (1), defined by

$$y_k(s)_q=\frac{\Delta}{\Delta x_{k-1}(s)}\frac{\Delta}{\Delta x_{k-2}(s)}\cdots\frac{\Delta}{\Delta x(s)}y(s)\equiv\Delta^{(k)}y(s)$$

also satisfies a difference equation of the hypergeometric type

$$\sigma(s)\frac{\Delta}{\Delta x_k(s-\frac{1}{2})}\left[\frac{\nabla y_k(s)_q}{\nabla x_k(s)}\right]+\tau_k(s)\frac{\Delta y_k(s)_q}{\Delta x_k(s)}+\mu_k y_k(s)_q=0\tag{3}$$

where $x_k(s) = x(s + \frac{k}{2})$ and [23, p 62, equation (3.1.29)]

$$\tau_k(s)=\frac{\sigma(s+k)-\sigma(s)+\tau(s+k)\Delta x(s+k-\frac{1}{2})}{\Delta x_{k-1}(s)}\qquad \mu_k=\lambda+\sum_{m=0}^{k-1}\frac{\Delta\tau_m(s)}{\Delta x_m(s)}.\tag{4}$$

It is important to notice that the above difference equations have polynomial solutions of the hypergeometric type iff $x(s)$ is a function of the form [7, 24]

$$x(s)=c_1(q)q^s+c_2(q)q^{-s}+c_3(q)=c_1(q)[q^s+q^{-s-\mu}]+c_3(q)\tag{5}$$

where c_1, c_2, c_3 and $q^\mu = \frac{c_1}{c_2}$ are constants which, in general, depend on q [23, 24]. For the above lattice, a straightforward calculation shows that $\tau_k(s)$ is a polynomial of first degree in $x_k(s)$ of the form (see, for example, [7])

$$\begin{gathered}\tau_k(s)=\tilde\tau_k'x_k(s)+\tilde\tau_k(0)\qquad \tilde\tau_k'=[2k]_q\frac{\tilde\sigma''}{2}+\alpha_q(2k)\tilde\tau'\\ \tilde\tau_k(0)=\frac{c_3\tilde\sigma''}{2}(2[k]_q-[2k]_q)+\tilde\sigma'(0)[k]_q+c_3\tau'(\alpha_q(k)-\alpha_q(2k))+\tilde\tau(0)\alpha_q(k)\end{gathered}\tag{6}$$

where the *q-numbers* $[k]_q$ and $\alpha_q(k)$ are defined by

$$[k]_q = \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \qquad \alpha_q(k) = \frac{q^{\frac{k}{2}} + q^{-\frac{k}{2}}}{2} \tag{7}$$

and $[n]_q!$ are the q-factorials $[n]_q! = [1]_q[2]_q \dots [n]_q$.

Both difference equations (1) and (3) can be rewritten in the symmetric form

$$\frac{\Delta}{\Delta x(s-\frac{1}{2})}\left[\sigma(s)\rho(s)\frac{\nabla y(s)}{\nabla x(s)}\right] + \lambda_n \rho(s) y(s) = 0$$

and

$$\frac{\Delta}{\Delta x_k(s-\frac{1}{2})}\left[\sigma(s)\rho_k(s)\frac{\nabla y_k(s)}{\nabla x_k(s)}\right] + \mu_k \rho_k(s) y_k(s) = 0$$

where $\rho(s)$ and $\rho_k(s)$ are the weight functions satisfying the Pearson-type difference equations

$$\frac{\Delta}{\Delta x(s-\frac{1}{2})}[\sigma(s)\rho(s)] = \tau(s)\rho(s) \qquad \frac{\Delta}{\Delta x_k(s-\frac{1}{2})}[\sigma(s)\rho_k(s)] = \tau_k(s)\rho_k(s) \tag{8}$$

respectively. In [23] it is shown that the polynomial solutions of (3) (and so the polynomial solutions of (1)) are determined by the q-analogue of the Rodrigues formula on the nonuniform lattices

$$\frac{\Delta}{\Delta x_{k-1}(s)} \cdots \frac{\Delta}{\Delta x(s)} P_n(x(s))_q \equiv \Delta^{(k)} P_n(x(s))_q = \frac{A_{n,k}B_n}{\rho_k(s)} \nabla_k^{(n)} \rho_n(s) \tag{9}$$

where

$$\begin{aligned} \nabla_k^{(n)} f(s) &= \frac{\nabla}{\nabla x_{k+1}(s)} \frac{\nabla}{\nabla x_{k+2}(s)} \cdots \frac{\nabla}{\nabla x_n(s)} f(s) \\ A_{n,k} &= \frac{[n]_q!}{[n-k]_q!} \prod_{m=0}^{k-1} \left\{ \alpha_q(n+m-1)\tilde{\tau}' + [n+m-1]_q \frac{\tilde{\sigma}''}{2} \right\}. \end{aligned} \tag{10}$$

Thus [23, p 66, equation (3.2.19)]

$$P_n(x(s))_q = \frac{B_n}{\rho(s)} \nabla^{(n)} \rho_n(s) \qquad \nabla^{(n)} \equiv \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)} \tag{11}$$

where $\rho_n(s) = \rho(s+n) \prod_{k=1}^{n} \sigma(s+k)$ and

$$\lambda_n = -[n]_q \left\{ \alpha_q(n-1)\tilde{\tau}' + [n-1]_q \frac{\tilde{\sigma}''}{2} \right\}. \tag{12}$$

In this paper we deal with orthogonal q-polynomials and functions. It can be proven [23], by using the difference equation of hypergeometric type (1), that if the boundary condition

$$\sigma(s)\rho(s)x^k(s-\tfrac{1}{2})\big|_{s=a,b} = 0 \qquad \forall k \geqslant 0 \tag{13}$$

holds, then the polynomials $P_n(s)_q$ are orthogonal, i.e.

$$\sum_{s=a}^{b-1} P_n(x(s))_q P_m(x(s))_q \rho(s) \Delta x(s-\tfrac{1}{2}) = \delta_{nm} d_n^2 \qquad s = a, a+1, \dots, b-1 \tag{14}$$

where $\rho(s)$ is a solution of the Pearson-type equation (8). In the special case of the linear exponential lattice $x(s) = q^s$ the above relation can be written in terms of the Jackson q-integral (see, for example, [15, 17]) $\int_{z_1}^{z_2} f(t)\,\mathrm{d}_q t$, defined by

$$\int_{z_1}^{z_2} f(t)\,\mathrm{d}_q t = \int_0^{z_2} f(t)\,\mathrm{d}_q t - \int_0^{z_1} f(t)\,\mathrm{d}_q t$$

where

$$\int_0^z f(t)\, \mathrm{d}_q t = z(1-q)\sum_{k=0}^{\infty} f(zq^k)q^k \qquad 0<q<1$$

as follows:

$$\int_{q^a}^{q^b} P_n(t)_q P_m(t)_q \omega(t)\, \mathrm{d}_q t = \delta_{nm} q^{1/2} d_n^2 \qquad t=q^s \quad \omega(t)\equiv\omega(q^t)=\rho(t). \tag{15}$$

Notice that the above boundary condition (13) is valid for $k=0$. Moreover, if we assume that a is finite, then (13) is fulfilled at $s=a$ providing that $\sigma(a)=0$ [23, section 3.3, p 70]. In the following we assume that this condition holds. The squared norm in (14) is given by [23, Chapter 3, section 3.7.2, p 104]

$$d_n^2 = (-1)^n A_{n,n} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s)\Delta x_n(s-\tfrac{1}{2}).$$

There is also a so-called continuous orthogonality. In fact, if there exists a contour Γ such that

$$\int_\Gamma \Delta[\rho(z)\sigma(z)x^k(z-\tfrac{1}{2})]\,\mathrm{d}z = 0 \qquad \forall k\geqslant 0 \tag{16}$$

then [23]

$$\int_\Gamma P_n(x(z))_q P_m(x(z))_q \rho(z)\Delta x(z-\tfrac{1}{2})\,\mathrm{d}z = 0 \qquad n\neq m.$$

A simple consequence of the orthogonality is the following three-term recurrence relation:

$$x(s)P_n(x(s))_q = \alpha_n P_{n+1}(x(s))_q + \beta_n P_n(x(s))_q + \gamma_n P_{n-1}(x(s))_q \tag{17}$$

where α_n, β_n and γ_n are constants. If $P_n(s)_q = a_n x^n(s) + b_n x^{n-1}(s) + \cdots$ then, using (17), we find

$$\alpha_n = \frac{a_n}{a_{n+1}} \qquad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \qquad \gamma_n = \frac{a_{n-1}}{a_n}\frac{d_n^2}{d_{n-1}^2}. \tag{18}$$

To obtain the explicit values of α_n, β_n we use the following lemma—interesting in its own right—that can be proven by induction.

Lemma 2.1.

$$\Delta^{(k)} x^n(s) = \frac{[n]_q!}{[n-k]_q!} x_k^{n-k}(s) + c_3\left(n\frac{[n-1]_q!}{[n-k-1]_q!} - (n-k)\frac{[n]_q!}{[n-k]_q!}\right) x_k^{n-k-1}(s) + \cdots.$$

In the case $k=n-1$, this becomes

$$\Delta^{(n-1)} x^n(s) = [n]_q! x_{n-1}(s) + c_3[n-1]_q!\left(n-[n]_q\right). \tag{19}$$

Now, using the Rodrigues formula (9) for $k=n-1$,

$$\Delta^{(n-1)} P_n(x(s))_q = \frac{A_{n,n-1}B_n}{\rho_{n-1}(s)}\nabla_{n-1}^{(n)}\rho_n(s) = \frac{A_{n,n-1}B_n}{\rho_{n-1}(s)}\frac{\nabla}{\nabla x_n(s)}\rho_n(s)$$

as well as the identities $\rho_n(s) = \rho_{n-1}(s+1)\sigma(s+1)$, $x_n(s) = x_{n-1}(s+\frac{1}{2})$ and the Pearson equation (8) for $\rho_{n-1}(s)$, we find that

$$\Delta^{(n-1)} P_n(x(s))_q = A_{n,n-1}B_n\tau_{n-1}(s).$$

Thus

$$a_n = \frac{A_{n,n-1}B_n\tilde{\tau}'_{n-1}}{[n]_q!} = B_n\prod_{k=0}^{n-1}\left\{\alpha_q(n+k-1)\tilde{\tau}' + [n+k-1]_q\frac{\tilde{\sigma}''}{2}\right\}$$

and

$$\frac{b_n}{a_n} = \frac{[n]_q \tilde{\tau}_{n-1}(0)}{\tilde{\tau}'_{n-1}} + c_3([n]_q - n).$$

So

$$\alpha_n = \frac{B_n}{B_{n+1}} \frac{\alpha_q(n-1)\tilde{\tau}' + [n-1]_q \frac{\tilde{\sigma}''}{2}}{(\alpha_q(2n-1)\tilde{\tau}' + [2n-1]_q \frac{\tilde{\sigma}''}{2})(\alpha_q(2n)\tilde{\tau}' + [2n]_q \frac{\tilde{\sigma}''}{2})}$$
$$= -\frac{B_n}{B_{n+1}} \frac{\lambda_n}{[n]_q} \frac{[2n]_q}{\lambda_{2n}} \frac{[2n+1]_q}{\lambda_{2n+1}}$$

and

$$\beta_n = \frac{[n]_q \tilde{\tau}_{n-1}(0)}{\tilde{\tau}'_{n-1}} - \frac{[n+1]_q \tilde{\tau}_n(0)}{\tilde{\tau}'_n} + c_3([n]_q + 1 - [n+1]_q).$$

Using the Rodrigues formula the following difference–recurrent relation follows [1,23]:

$$\sigma(s)\frac{\nabla P_n(x(s))_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q \tau'_n}\left[\tau_n(s) P_n(x(s))_q - \frac{B_n}{B_{n+1}} P_{n+1}(x(s))_q\right]$$

where $\tau_n(s)$ is given by (6), where the identity $\tilde{\tau}'_n = -\frac{\lambda_{2n+1}}{[2n+1]_q}$ has been used.

Then, using the explicit expression for the coefficient α_n, we find

$$\sigma(s)\frac{\nabla P_n(x(s))_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau'_n} P_n(x(s))_q - \frac{\alpha_n \lambda_{2n}}{[2n]_q} P_{n+1}(x(s))_q. \tag{20}$$

This equation defines a raising operator in terms of the backward difference in the sense that we can obtain the polynomial P_{n+1} of degree $n+1$ from the lower-degree polynomial P_n.

From the above equation and using the identity $\nabla = \Delta - \nabla\Delta$, the second-order difference equation and the three-term recurrence relation we find the [1] lowering-type operator

$$\left[\sigma(s) + \tau(s)\Delta x\left(s - \frac{1}{2}\right)\right]\frac{\Delta P_n(x(s))_q}{\Delta x(s)} = \frac{\gamma_n \lambda_{2n}}{[2n]_q} P_{n-1}(x(s))_q$$
$$+ \left[\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau'_n} - \lambda_n \Delta x\left(s - \frac{1}{2}\right) - \frac{\lambda_{2n}}{[2n]_q}(x(s) - \beta_n)\right] P_n(x(s))_q. \tag{21}$$

The most general polynomial solution of the q-hypergeometric equation (1) corresponds to the case

$$\sigma(s) = A\prod_{i=1}^{4}[s - s_i]_q = Cq^{-2s}\prod_{i=1}^{4}(q^s - q^{s_i}) \qquad A, C, \text{ not vanishing constants} \tag{22}$$

and has the form [24]

$$P_n(s)_q = D_n\, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{2\mu+n-1+\sum_{i=1}^4 s_i}, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{matrix}; q, q\right) \tag{23}$$

where D_n is a normalizing constant and the basic hypergeometric series ${}_p\phi_q$ are defined by [17]

$${}_r\phi_p\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_p \end{matrix}; q, z\right) = \sum_{k=0}^{\infty}\frac{(a_1;q)_k \ldots (a_r;q)_k}{(b_1;q)_k \ldots (b_p;q)_k}\frac{z^k}{(q;q)_k}\left[(-1)^k q^{\frac{k}{2}(k-1)}\right]^{p-r+1}$$

and

$$(a;q)_k = \prod_{m=0}^{k-1}(1 - aq^m) \tag{24}$$

is the q-analogue of the Pochhammer symbol. Instances of such polynomials are the Askey–Wilson polynomials, the q-Racah polynomials and big q-Jacobi polynomials among others [17,24].

3. The orthonormal functions on nonuniform lattices

In this section we introduce a set of orthonormal functions which are orthogonal with respect to the unit weight [6, 27]

$$\varphi_n(s) = \sqrt{\rho(s)/d_n^2}\, P_n(x(s))_q. \tag{25}$$

For example, for the case of discrete orthogonality we have

$$\sum_{s_i=a}^{b-1} \varphi_n(s_i)\varphi_m(s_i)\Delta x(s_i - \tfrac{1}{2}) = \delta_{nm}.$$

Next, we will establish several important properties of such functions which generalize, to the nonuniform lattices, the ones given in [18]. In the following we use the notation $\Theta(s) = \sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})$.

First of all, inserting (25) into (1), (17), (20), (21) we obtain that they satisfy the difference equation

$$\begin{aligned}\sqrt{\Theta(s)\sigma(s+1)}\frac{1}{\Delta x(s)}\varphi_n(s+1) + \sqrt{\Theta(s-1)\sigma(s)}\frac{1}{\nabla x(s)}\varphi_n(s-1)\\ -\left(\frac{\Theta(s)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)}\right)\varphi_n(s) + \lambda_n \Delta x\left(s - \frac{1}{2}\right)\varphi_n(s) = 0\end{aligned} \tag{26}$$

the three-term recurrence relation

$$\alpha_n \frac{d_{n+1}}{d_n}\varphi_{n+1}(s) + \gamma_n \frac{d_{n-1}}{d_n}\varphi_{n-1}(s) + (\beta_n - x(s))\varphi_n(s) = 0 \tag{27}$$

the raising-type formula

$$L^+(s,n)\varphi_n(s) = \alpha_n \frac{\lambda_{2n}}{[2n]_q}\frac{d_{n+1}}{d_n}\varphi_{n+1}(s) \tag{28}$$

and the lowering-type formula

$$L^-(s,n)\varphi_n(s) = \gamma_n \frac{\lambda_{2n}}{[2n]_q}\frac{d_{n-1}}{d_n}\varphi_{n-1}(s) \tag{29}$$

where the raising-type operator $L^+(s,n)$ and the lowering-type operator $L^-(s,n)$ are given by

$$L^+(s,n) \equiv \left[\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} - \frac{\sigma(s)}{\nabla x(s)}\right] I + \sqrt{\Theta(s-1)\sigma(s)}\frac{1}{\nabla x(s)}E^- \tag{30}$$

and

$$\begin{aligned}L^-(s,n) \equiv \left[-\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} + \lambda_n \Delta x\left(s - \frac{1}{2}\right) + \frac{\lambda_{2n}}{[2n]_q}(x(s) - \beta_n) - \frac{\Theta(s)}{\Delta x(s)}\right] I\\ + \sqrt{\Theta(s)\sigma(s+1)}\frac{1}{\Delta x(s)}E^+\end{aligned} \tag{31}$$

respectively. In the above formulae $E^- f(s) = f(s-1)$, $E^+ f(s) = f(s+1)$ and I is the identity operator.

Note that the last two formulae have a remarkable property of giving all the solutions $\varphi_n(s)$. In fact, from (31) setting $n = 0$ and taking into account that $\varphi_{-1}(s) \equiv 0$ we can obtain $\varphi_0(s)$. Then, substituting the obtained function in (30), we can find all the functions $\varphi_1(s), \ldots, \varphi_n(s), \ldots$.

Proposition 3.1. *The raising and lowering operators (30) and (31) are mutually adjoint.*

Proof. The proof is straightforward. Using the boundary condition and after some calculations we obtain, in the case of discrete orthogonality, the expression

$$\sum_{s_i=a}^{b-1} \varphi_{n+1}(s_i)\left[\frac{[2n]_q}{\lambda_{2n}} L^+(s_i,n)\varphi_n(s_i)\right]\Delta x\left(s_i-\frac{1}{2}\right)$$
$$=\sum_{s_i=a}^{b-1}\left[\frac{[2n+2]_q}{\lambda_{2n+2}} L^-(s_i,n+1)\varphi_{n+1}(s_i)\right]\varphi_n(s_i)\Delta x\left(s_i-\frac{1}{2}\right)=\alpha_n\frac{d_{n+1}}{d_n}.$$

The other cases can be dealt with in an analogous way. □

Proposition 3.2. *The operator corresponding to the eigenvalue λ_n in (26) is self-adjoint.*

Proof. Again, we prove the result in the case of discrete orthogonality. Using the orthogonality conditions $\sigma(a)\rho(a)=\sigma(b)\rho(b)=0$ (which is a consequence of (13)), we can write

$$\sum_{s_i=a}^{b-1}\varphi_n(s_i)\sqrt{\Theta(s_i-1)\sigma(s_i)}\frac{1}{\nabla x(s_i)}\varphi_l(s_i-1)\Delta x\left(s_i-\frac{1}{2}\right)$$
$$=\sum_{s_i'=a-1}^{b-2}\varphi_n(s_i'+1)\sqrt{\Theta(s_i')\sigma(s_i'+1)}\frac{1}{\nabla x(s_i'+1)}\varphi_l(s_i')\Delta x\left(s_i'+\frac{1}{2}\right)$$
$$=\sum_{s_i=a}^{b-1}\varphi_n(s_i+1)\sqrt{\Theta(s_i)\sigma(s_i+1)}\frac{1}{\nabla x(s_i+1)}\varphi_l(s_i)\Delta x\left(s_i+\frac{1}{2}\right)$$
$$+\varphi_n(a)\sqrt{\Theta(a-1)\sigma(a)}\frac{1}{\nabla x(a)}\varphi_l(a-1)\Delta x\left(a-\frac{1}{2}\right)$$
$$-\varphi_n(b)\sqrt{\Theta(b-1)\sigma(b)}\frac{1}{\nabla x(b)}\varphi_l(b-1)\Delta x\left(b-\frac{1}{2}\right)$$

where in the last two sums we first take the operations Δ and ∇, and then substitute the corresponding value: for example, $\Delta x(a)=x(a+1)-x(a)$.

Now, we use the fact that $\varphi_n(s)=\sqrt{\rho(s)/d_n^2}P_n(x(s))_q$, as well as the boundary conditions $\sigma(a)\rho(a)=\sigma(b)\rho(b)=0$, so

$$\sqrt{\Theta(a-1)\sigma(a)}\,\varphi_n(a)\varphi_l(a-1)=\sqrt{\Theta(b-1)\sigma(b)}\varphi_n(b)\varphi_l(b-1)=0.$$

The other terms can be transformed in a similar way. All these yield the expression

$$\sum_{s_i=a}^{b-1}\varphi_l(s_i)\left\{\sqrt{\Theta(s_i)\sigma(s_i+1)}\frac{1}{\Delta x(s_i)}\varphi_n(s_i+1)\Delta x\left(s_i+\frac{1}{2}\right)\right.$$
$$\left.+\sqrt{\Theta(s_i-1)\sigma(s_i)}\frac{1}{\nabla x(s_i)}\varphi_n(s_i-1)\Delta x\left(s_i-\frac{1}{2}\right)\right\}$$
$$=\sum_{s_i=a}^{b-1}\varphi_n(s_i)\left\{\sqrt{\Theta(s_i)\sigma(s_i+1)}\frac{1}{\Delta x(s_i)}\varphi_l(s_i+1)\Delta x\left(s_i+\frac{1}{2}\right)\right.$$
$$\left.+\sqrt{\Theta(s_i-1)\sigma(s_i)}\frac{1}{\nabla x(s_i)}\varphi_l(s_i-1)\Delta x\left(s_i-\frac{1}{2}\right)\right\}$$

from which the proposition easily follows. □

4. Factorization of difference equation of hypergeometric type on the nonuniform lattice

We define from (26) the following operator:

$$H(s,n) \equiv \sqrt{\Theta(s-1)\sigma(s)}\frac{1}{\nabla x(s)}E^- + \sqrt{\Theta(s)\sigma(s+1)}\frac{1}{\Delta x(s)}E^+ - \left(\frac{\Theta(s)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} - \lambda_n \Delta x\left(s-\frac{1}{2}\right)\right)I.$$

Clearly, the orthonormal functions satisfy

$$H(s,n)\varphi_n(s) = 0.$$

Let us rewrite the raising and lowering operators in the following way:

$$L^+(s,n) = u(s,n)I + \sqrt{\Theta(s-1)\sigma(s)}\frac{1}{\nabla x(s)}E^-$$

$$L^-(s,n) = v(s,n)I + \sqrt{\Theta(s)\sigma(s+1)}\frac{1}{\Delta x(s)}E^+$$

where, as before, $\Theta(s) = \sigma(s) + \tau(s)\Delta x(s-\frac{1}{2})$, and

$$u(s,n) = \frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} - \frac{\sigma(s)}{\nabla x(s)}$$

$$v(s,n) = -\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} + \lambda_n \Delta x\left(s-\frac{1}{2}\right) + \frac{\lambda_{2n}}{[2n]_q}(x(s)-\beta_n) - \frac{\Theta(s)}{\Delta x(s)}.$$

Proposition 4.1. *The functions $u(s,n)$ and $v(s,n)$ satisfy $u(s+1,n) = v(s,n+1)$ or, equivalently, $u(s+1,n-1) = v(s,n)$.*

The proof of the above proposition is straightforward but cumbersome. We include it in appendix. If we now calculate

$$L^-(s,n+1)L^+(s,n) = v(s,n+1)u(s,n) + \Theta(s)\sigma(s+1)\left(\frac{1}{\Delta x(s)}\right)^2 + u(s+1,n)\left\{\sqrt{\Theta(s-1)\sigma(s)}\frac{1}{\nabla x(s)}E^- + \sqrt{\Theta(s)\sigma(s+1)}\frac{1}{\Delta x(s)}E^+\right\}$$

and substitute the values for $u(s,n)$, $v(s,n)$ and $H(s,n)$ we get

$$L^-(s,n+1)L^+(s,n) = h^{\mp}(n)I + u(s+1,n)H(s,n)$$

where the function

$$h^{\mp}(n) = \left(\frac{\lambda_n}{[n]_q}\frac{\tau_n(s+1)}{\tau_n'} - \frac{\sigma(s+1)}{\nabla x(s+1)}\right)\left(\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} - \lambda_n\Delta x\left(s-\frac{1}{2}\right)\right) + \frac{\lambda_n}{[n]_q}\frac{\tau_n(s+1)}{\tau_n'}\frac{\Theta(s)}{\Delta x(s)}$$

is independent of s. In fact, applying the last equality to the orthonormal function $\varphi_n(s)$ and taking into account (28) and (29),

$$h^{\mp}(n) = \frac{\lambda_{2n}}{[2n]_q}\frac{\lambda_{2n+2}}{[2n+2]_q}\alpha_n\gamma_{n+1}.$$

Similarly,

$$L^+(s,n-1)L^-(s,n) = h^{\pm}(n)I + u(s,n-1)H(s,n)$$

where

$$h^{\pm}(n) = \left(-\frac{\lambda_n}{[n]_q}\frac{\tau_n(s-1)}{\tau_n'} + \frac{\lambda_{2n}}{[2n]_q}(x(s-1)-\beta_n) + \lambda_n \Delta x\left(s-\frac{3}{2}\right)\right)$$
$$\times\left(-\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} + \frac{\lambda_{2n}}{[2n]_q}(x(s)-\beta_n) + \frac{\sigma(s)}{\nabla x(s)}\right)$$
$$= \left(-\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} + \frac{\lambda_{2n}}{[2n]_q}(x(s)-\beta_n)\right)\left(\frac{\Theta(s-1)}{\Delta x(s-1)}\right)$$

is independent of s. Furthermore, applying the last expression to the functions $\varphi_n(s)$, and taking into account (28) and (29), we obtain

$$h^{\pm}(n) = \frac{\lambda_{2n-2}}{[2n-2]_q}\frac{\lambda_{2n}}{[2n]_q}\alpha_{n-1}\gamma_n.$$

Remark. Notice that $h^{\pm}(n+1) = h^{\mp}(n)$.

All the above results lead us to our main theorem.

Theorem 4.1. *The operator $H(s,n)$, corresponding to the hypergeometric difference equation for orthonormal functions $\varphi_n(s)$ admits the following factorization—usually called the Infeld–Hull-type factorization:*

$$u(s+1,n)H(s,n) = L^-(s,n+1)L^+(s,n) - h^{\mp}(n)I \tag{32}$$

and

$$u(s,n)H(s,n+1) = L^+(s,n)L^-(s,n+1) - h^{\mp}(n)I \tag{33}$$

respectively.

Remark. Substituting in the above formulae the expression $x(s) = s$ we obtain the corresponding results for the uniform lattice cases (Hahn, Kravchuk, Meixner and Charlier), considered before by several authors (see, for example, [6, 18, 27] and by taking appropriate limits (see, for example, [17, 23]), we can recover the classical continuous case (Jacobi, Laguerre and Hermite).

5. Applications to some q-normalized orthogonal functions

For the sake of completeness we apply the above results to several families of orthogonal q-polynomials and their corresponding orthonormal q-functions that are of interest and appear in several branches of mathematical physics. They are the Al-Salam and Carlitz polynomials I and II, the big q-Jacobi polynomials, the dual q-Hahn polynomials, the continuous q-Hermite and the celebrated q-Askey–Wilson polynomials.

The main data for these polynomials are taken from [17], except for the case of dual q-Hahn polynomials [3]. They can be obtained from the general formulae given in section 2.

Finally, let us point out that similar factorization formulae have been obtained by other authors: for example, Miller in [21] considered the polynomials on the linear exponential lattice and Bangerezako studied the Askey–Wilson case. Our main aim in this section is to show how our general formulae lead, in a very easy way, to the needed factorization formulae of several families for normalized functions—not polynomials.

5.1. The Al-Salam and Carlitz functions I and II

The Al-Salam and Carlitz polynomials I (and II) appear in certain models of q-harmonic oscillator: see, for example, [5, 8, 9, 22]. They are polynomials on the exponential lattice $x(s) = q^s \equiv x$, defined [17] by

$$U_n^{(a)}(x; q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; \frac{qx}{a} \right)$$

and constitute an orthogonal family with the orthogonality relation (15)

$$\int_a^1 U_n^{(a)}(x; q) U_m^{(a)}(x; q) \omega(x) \, \mathrm{d}_q x = d_n^2 \delta_{nm}$$

where

$$\omega(x) = (qx, a^{-1}qx; q)_\infty \qquad \text{and} \qquad d_n^2 = (-a)^n (1-q)(q; q)_n (q, a, a^{-1}q; q)_\infty q^{\binom{n}{2}}.$$

As usual, $(a_1, \ldots, a_p; q)_n = (a_1; q)_n \ldots (a_p; q)_n$, and $(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$.

They satisfy a difference equation of the form (1) where

$$\sigma(x) = (x-1)(x-a) \qquad \tau(x) = \tilde{\tau}(x) = \tau' x + \tau(0)$$

with

$$\tau' = \frac{q^{1/2}}{1-q} \qquad \tau(0) = q^{1/2} \frac{1+a}{q-1}.$$

The eigenvalues λ_n and the coefficients of the TTRR are given by

$$\lambda_n = [n]_q \frac{q^{1-n/2}}{q-1} \qquad \text{and} \qquad \alpha_n = 1 \qquad \beta_n = (1+a)q^n \qquad \gamma_n = aq^{n-1}(q^n - 1)$$

respectively. In this case we have

$$\tilde{\sigma}'' = 1 \qquad \tilde{\sigma}'(0) = -\frac{a+1}{2} \qquad \tilde{\sigma}(0) = a \qquad \tau_n' = \frac{q^{\frac{1}{2}-n}}{1-q} \qquad \tau_n(0) = q^{\frac{1-n}{2}} \frac{a+1}{q-1}.$$

The corresponding normalized functions (25) are

$$\varphi_n(x) = \sqrt{\frac{(qx, a^{-1}qx; q)_\infty (-a)^n q^{\binom{n}{2}}}{(1-q)(q; q)_n (q, a, q/a; q)_\infty}} \, {}_2\varphi_1 \left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; \frac{qx}{a} \right).$$

Defining now the Hamiltonian for these functions $\varphi_n(x)$

$$H(x, n) = \frac{\sqrt{a(x-1)(x-a)}}{x(1-q^{-1})} E^- + \frac{\sqrt{a(qx-1)(qx-a)}}{x(q-1)} E^+ + \left(\frac{q^{1-n}}{1-q} x + \frac{q(a+1)}{q-1} - \frac{[2]_q}{k_q} x^{-1} \right) I$$

and using that $u(x, n) = \frac{aq}{1-q} x^{-1}$, $v(x, n) = u(qx, n-1) = \frac{a}{1-q} x^{-1}$, thus

$$L^+(x, n) = u(x, n) I + q \frac{\sqrt{a(x-1)(x-a)}}{x(q-1)} E^- \qquad \text{where} \quad E^- f(x) = f(q^{-1}x)$$

and

$$L^-(x, n) = v(x, n) I + \frac{\sqrt{a(qx-1)(qx-a)}}{x(q-1)} E^+ \qquad \text{where} \quad E^+ f(x) = f(qx)$$

we have

$$L^-(x, n+1) L^+(x, n) = \frac{aq^{1-n}(q^{n+1} - 1)}{(q-1)^2} I + v(x, n+1) H(x, n)$$

and

$$L^{+}(x, n-1)L^{-}(x, n) = \frac{aq^{2-n}(q^n - 1)}{(q-1)^2} I + u(x, n-1)H(x, n)$$

which give the factorization formulae for the Al-Salam and Carlitz functions I. If we now take into account that (see [17, p 115])

$$V_n^{(a)}(x; q) = U_n^{(a)}(x; q^{-1})$$

then the factorization for the Al-Salam and Carlitz functions II

$$\varphi_n(s) = q^{\binom{s}{2}} \sqrt{\frac{a^{s+n}(aq; q)_\infty q^{\binom{n+1}{2}}}{(q, aq; q)_s (1-q)(q; q)_n}}\, {}_2\phi_0 \left(\begin{matrix} q^{-n}, x \\ - \end{matrix} \middle| q; \frac{q^n}{a} \right)$$

follows from the factorization for the Al-Salam and Carlitz functions I simply by changing q to q^{-1}.

5.2. *The big q-Jacobi functions*

Now we consider the most general family of q-polynomials on the exponential lattice, the so-called big q-Jacobi polynomials, that appear in the representation theory of the quantum algebras [31]. They were introduced by Hahn in 1949 and are defined [17] by

$$P_n(x; a, b, c; q) = \frac{(aq; q)_n (cq; q)_n}{(abq^{n+1}; q)_n}\, {}_3\phi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q; q \right) \qquad x(s) = q^s \equiv x.$$

They constitute an orthogonal family, i.e.

$$\int_{cq}^{aq} \omega(x) P_n(x; a, b, c; q) P_n(x; a, b, c; q)\, \mathrm{d}_q x = d_n^2 \delta_{nm}$$

where

$$\omega(x) = \frac{(a^{-1}x; q)_\infty (c^{-1}x; q)_\infty}{(x; q)_\infty (bc^{-1}x; q)_\infty}$$

$$d_n^2 = \frac{aq(1-q)(q, c/a, aq/c, abq^2; q)_\infty}{(aq, bq, cq, abq/c; q)_\infty} \frac{(1-abq)(q, bq, abq/c; q)_n (-ac)^{-n} q^{-\binom{n}{2}}}{(abq, abq^{n+1}, abq^{n+1})_n}.$$

They satisfy the difference equation (1) with

$$\sigma(x) = q^{-1}(x - aq)(x - cq) \qquad \text{and} \qquad \tau(x) = \tilde{\tau}(x) = \tau' x + \tau(0)$$

where

$$\tau' = \frac{1 - abq^2}{(1-q)q^{1/2}} \qquad \text{and} \qquad \tau(0) = q^{1/2} \frac{a(bq-1) + c(aq-1)}{1-q}$$

and

$$\lambda_n = -q^{-n/2} [n]_q \frac{1 - abq^{n+1}}{1-q}.$$

They satisfy a TTRR, whose coefficients are

$$\alpha_n = 1 \qquad \beta_n = 1 - A_n - C_n \qquad \gamma_n = C_n A_{n-1}$$

where

$$A_n = \frac{(1 - aq^{n+1})(1 - cq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})}$$

$$C_n = -acq^{n+1} \frac{(1 - q^n)(1 - bq^n)(1 - abc^{-1}q^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}.$$

Also, we have

$$\tilde{\sigma}'' = \frac{1+abq^2}{q} \qquad \tilde{\sigma}'(0) = -\frac{abq+acq+a+c}{2} \qquad \tilde{\sigma}(0) = acq$$

$$\tau_n' = \frac{q^{-n}-abq^{n+2}}{q^{1/2}(1-q)} \qquad \tau_n(0) = q^{\frac{1-n}{2}}\frac{a(bq^{1+n}-1)+c(aq^{1+n}-1)}{1-q}.$$

The normalized big q-Jacobi functions are defined by

$$\varphi_n(s) = \sqrt{\frac{(x/a, x/c; q)_\infty (aq, bq, abq/c; q)_\infty (abq, aq, aq, cq, cq; q)_n (-ac)^n}{(x, bx/c, c/a, aq/c, abq^2; q)_\infty (1-q)aq(1-abq)(q, bq, abq/c; q)_n}} \times {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}\,\middle|\, q; q\right).$$

The corresponding Hamiltonian is

$$H(x,n) = \frac{\sqrt{a(x-q)(x-aq)(x-cq)(bx-cq)}}{x(q-1)}E^- + q\frac{\sqrt{a(x-1)(x-a)(x-c)(bx-c)}}{x(q-1)}E^+ + \left(\frac{1+abq^{2n+1}}{q^n(1-q)}x - \frac{q(a+ab+c+ac)}{1-q} + \frac{acq(q+1)}{1-q}x^{-1}\right)I.$$

Furthermore,

$$u(x,n) = \frac{abq^{n+1}}{1-q}x + D_n - \frac{acq^2}{q-1}x^{-1}$$

$$v(x,n) = \frac{abq^{n+1}}{1-q}x + D_{n-1} - \frac{acq}{q-1}x^{-1}$$

where

$$D_n = \frac{ab(ab+ac+a+c)q^{2n+3} - a(b+c+ab+bc)q^{n+2}}{(1-abq^{2n+2})(1-q)}$$

thus

$$L^+(x,n) = u(x,n)I + \frac{\sqrt{a(x-q)(x-aq)(x-cq)(bx-cq)}}{x(q-1)}E^- \quad \text{where} \quad E^- f(x) = f(q^{-1}x)$$

and

$$L^-(x,n) = v(x,n)I + q\frac{\sqrt{a(x-1)(x-a)(x-c)(bx-c)}}{x(q-1)}E^+ \quad \text{where} \quad E^+ f(x) = f(qx)$$

so

$$L^-(x,n+1)L^+(x,n) = \delta_{n+1}\gamma_{n+1}I + v(x,n+1)H(x,n)$$

$$L^+(x,n-1)L^-(x,n) = \delta_n\gamma_n I + u(x,n-1)H(x,n)$$

where

$$\delta_n = \frac{(1-abq^{2n-1})(1-abq^{2n+1})}{q^{2n-1}(q-1)^2}.$$

The above formulae are the factorization formulae for the family of the big q-Jacobi normalized functions.

Since all discrete q-polynomials on the exponential lattice $x(s) = c_1 q^s + c_3$—the so-called q-Hahn class—can be obtained from the big q-Jacobi polynomials by a certain limit process (see, for example, [2, 17]), then from the above formulae we can obtain the factorization formulae for the all other cases in the q-Hahn tableau. Of special interest are the q-Hahn polynomials and the big q-Laguerre polynomials, which are particular cases of the big q-Jacobi polynomials when $c = q^{-N-1}$, $N = 1, 2, \ldots$, and $c = 0$, respectively.

5.3. The q-dual-Hahn functions

In this section we will deal with the q-dual Hahn polynomials, introduced in [3,24] and closely related with the Clebsch–Gordan coefficients of the q-algebras $SU_q(2)$ and $SU_q(1,1)$ [3]. They are defined on the lattice $x(s) = [s]_q[s+1]_q$ by

$$W_n^c(x(s); a, b)_q = \frac{(-1)^n (q^{a-b+1}; q)_n (q^{a+c+1}; q)_n}{q^{n/2(3a-b+c+1+n)} k_q^n (q; q)_n} \, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{a+c+1} \end{matrix} \,\middle|\, q; q \right)$$

and satisfy a discrete orthogonality (14) with respect to the weight function

$$\rho(s) = \frac{q^{\frac{1}{2}((b-1)^2 - (2s-1)(a+c))}}{(1-q)^{2(a+c-b)+1}} \frac{(q^{s-a+1}, q^{s-c+1}, q^{s+b+1}, q^{b-s}; q)_\infty}{(q, q, q^{s+a+1}, q^{s+c+1}; q)_\infty}$$

where $-\frac{1}{2} \leqslant a < b - 1$, $|c| < a + 1$, and for this weight function the norm is

$$d_n^2 = \frac{q^{\frac{1}{4}(-4ab-4bc+6a+6c-8b+6+4n(a+c-2b)-n^2+17n+2b^2)}}{(1-q)^{2(a+c-b+1)+3n}} \frac{(q^{b-c-n}, q^{b-a-n}; q)_\infty}{[n]_q!(q, q^{a+c+n+1}; q)_\infty}.$$

These polynomials satisfy a TTRR (17) with

$$\alpha_n = 1$$

$$\beta_n = q^{\frac{1}{2}(2n-b+c+1)}[b-a-n+1]_q[a+c+n+1]_q + q^{\frac{1}{2}(2n+2a+c-b+1)}[n]_q[b-c-n]_q + [a]_q[a+1]_q$$

$$\gamma_n = q^{2n+c+a-b}[a+c+n]_q[b-a-n]_q[b-c-n]_q[n]_q$$

and the second-order difference equation (1), whose eigenvalues are $\lambda_n = [n]_q q^{\frac{1}{2}-\frac{n}{2}}$ and

$$\sigma(s) = q^{\frac{1}{2}(s+c+a-b+2)}[s-a]_q[s+b]_q[s-c]_q \qquad \text{and} \qquad \tau(x) = \tilde{\tau}(x) = \tau' x + \tau(0)$$

with $\tau' = -1$ and $\tau(0) = q^{\frac{1}{2}(a-b+c+1)}[a+1]_q[b-c-1]_q + q^{\frac{1}{2}(c-b+1)}[b]_q[c]_q$.

We also need the values

$$\tilde{\sigma}'' = k_q \qquad \tilde{\sigma}'(0) = \frac{1}{2k_q}(2[2]_q - q^{\frac{1}{2}-b} - q^{\frac{1}{2}+a} - q^{\frac{3}{2}+a+c-b} - q^{\frac{1}{2}+c})$$

$$\tilde{\sigma}(0) = \frac{1}{2k_q^3}(2q^{1+a-b} + q^{-1} + q + 2q^{1+c-b} + 2q^{1+a+c} - (1+q)(q^{-b} + q^a + q^c + q^{1+a+c-b}))$$

$$\tau_n' = -q^{-n}$$

$$\tau_n(0) = q^{\frac{1}{2}(c-b-n+1)}\left[c + \frac{n}{2}\right]_q\left[b - \frac{n}{2}\right]_q + q^{\frac{1}{2}(a+c-b+1-\frac{n}{2})}\left[a + \frac{n}{2} + 1\right]_q[b-c-n-1]_q.$$

In this case, the Hamiltonian, associated with the q-dual Hahn normalized functions $\sqrt{\rho(s)/d_n^2}\,W_n^c(x(s); a, b)_q$, is

$$H(s,n) = q^{\frac{1}{2}(c+a-b+2)} \frac{\sqrt{([s+1]_q^2 - [a]_q^2)([b]_q^2 - [s+1]_q^2)([s+1]_q^2 - [c]_q^2)}}{[2s+2]_q} E^+$$

$$+q^{\frac{1}{2}(c+a-b+2)}\frac{\sqrt{([s]_q^2-[a]_q^2)([b]_q^2-[s]_q^2)([s]_q^2-[c]_q^2)}}{[2s]_q}E^-$$
$$-q^{\frac{1}{2}-\frac{n}{2}}[n]_q[2s+1]_qI+q^{\frac{1}{2}(c+a-b+2)}$$
$$\times\left(\frac{[s-a]_q[s+b]_q+[s-c]_q}{[2s]_q}-\frac{[s+1-a]_q[s+1+b]_q[s+1-c]_q}{[2s+2]_q}\right)I$$

where $E^+f(s)=f(s+1)$ and $E^-f(s)=f(s-1)$. Then, using that

$$u(s,n)=q^{\frac{1}{2}-\frac{n}{2}}x(s+n/2)-q^{\frac{1}{2}+\frac{n}{2}}\left(q^{\frac{1}{2}(c-b-n+1)}\left[c+\frac{n}{2}\right]_q\left[b-\frac{n}{2}\right]_q\right.$$
$$\left.+q^{\frac{1}{2}(a+c-b+1-\frac{n}{2})}\left[a+\frac{n}{2}+1\right]_q[b-c-n-1]_q\right)$$
$$-q^{\frac{1}{2}(s+c+a-b+2)}\frac{[s-a]_q[s+b]_q[s-c]_q}{[2s]_q}$$

and taking into account that $v(s,n)=u(s+1,n-1)$, we find

$$L^+(s,n)=u(s,n)I+q^{\frac{1}{2}(c+a-b+2)}\frac{\sqrt{([s]_q^2-[a]_q^2)([b]_q^2-[s]_q^2)([s]_q^2-[c]_q^2)}}{[2s]_q}E^-$$

and

$$L^-(s,n)=v(s,n)I+q^{\frac{1}{2}(c+a-b+2)}\frac{\sqrt{([s+1]_q^2-[a]_q^2)([b]_q^2-[s+1]_q^2)([s+1]_q^2-[c]_q^2)}}{[2s+2]_q}E^+.$$

Thus

$$L^-(s,n+1)L^+(s,n)=q^{-2n}\gamma_{n+1}I+v(s,n+1)H(s,n)$$

and

$$L^+(s,n-1)L^-(s,n)=q^{-2n+2}\gamma_nI+u(s,n-1)H(s,n)$$

are the factorization formulae for the q-dual Hahn normalized functions.

5.4. *The Askey–Wilson functions*

Finally, we consider the family of Askey–Wilson polynomials. They are polynomials on the lattice $x(s)=\frac{1}{2}(q^s+q^{-s})\equiv x$, defined by [17]

$$p_n(x(s);a,b,c,d)=\frac{(ab;q)_n(ac;q)_n(ad;q)_n}{a^n}{}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,ae^{-i\theta},ae^{i\theta}\\ab,ac,ad\end{matrix}\middle|q;q\right)$$

i.e. they correspond to the general case (23) when $q^{s_1}=a$, $q^{s_2}=b$, $q^{s_3}=c$, $q^{s_4}=d$. Their orthogonality relation is of the form

$$\int_{-1}^{1}\omega(x)p_n(x;a,b,c,d)p_m(x;a,b,c,d)\sqrt{1-x^2}\kappa_q\,dx=\delta_{nm}d_n^2\qquad q^s=e^{i\theta}\quad x=\cos\theta$$

where

$$\omega(x)=\frac{h(x,1)h(x,-1)h(x,q^{\frac{1}{2}})h(x,-q^{\frac{1}{2}})}{2\pi\kappa_q(1-x^2)h(x,a)h(x,b)h(x,c)h(x,d)}\qquad h(x,\alpha)=\prod_{k=0}^{\infty}[1-2\alpha xq^k+\alpha^2q^{2k}]$$

and the norm is given by

$$d_n^2=\frac{(abcdq^{n-1};q)_n(abcdq^{2n};q)_\infty}{(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}.$$

The Askey–Wilson polynomials satisfy the difference equation (1) with

$$\sigma(s) = -q^{-2s+1/2}\kappa_q^2(q^s-a)(q^s-b)(q^s-c)(q^s-d) \qquad \kappa_q = (q^{\frac{1}{2}}-q^{-\frac{1}{2}})$$

and $\tau(x) = \tilde{\tau}(x) = \tau' x + \tau(0)$, where

$$\tau' = 4(q-1)(1-abcd) \qquad \tau(0) = 2(1-q)(a+b+c+d-abc-abd-acd-bcd).$$

Furthermore, they satisfy the TTRR (17) with coefficients

$$\alpha_n = 1 \qquad \beta_n = \frac{a+a^{-1}-(A_n+C_n)}{2} \qquad \gamma_n = \frac{C_n A_{n-1}}{4}$$

where A_n, C_n are defined by

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}$$
$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}$$

and whose eigenvalues are $\lambda_n = 4q^{-n+1}(1-q^n)(1-abcdq^{n-1})$. In addition, we have

$$\tilde{\sigma}'' = -4(q-1)^2(1+abcd)q^{-1/2}$$
$$\tilde{\sigma}'(0) = (q-1)^2(a+b+c+d+abc+abd+acd+bcd)q^{-1/2}$$
$$\tilde{\sigma}(0) = (q-1)^2(1-ab-ac-ad-bc-bd-cd+abcd)q^{-1/2}$$
$$\tau_n' = 4q^{-n}(q-1)(1-abcdq^{2n})$$
$$\tau_n(0) = 2(q-1)(-a-b-c-d+(abc+abd+acd+bcd)q^n)q^{-n/2}.$$

Defining now the normalized functions (see (15)) $\sqrt{\omega(x)/d_n^2}\,p_n(x;a,b,c,d)$, the corresponding Hamiltonian $H(s,n)$ is

$$H(s,n) = \frac{2q^{3/2}}{[2s-1]_q}G(s,a,b,c,d)E^- + \frac{2q^{3/2}}{[2s+1]_q}G(s+1,a,b,c,d)E^+$$
$$+2\left(q^{-2s+1/2}\frac{\prod_{i=1}^4(1-q^{s_i+s})}{[2s+1]_q} + q^{-2s+1/2}\frac{\prod_{i=1}^4(q^s-q^{s_i})}{[2s-1]_q}\right.$$
$$\left.+q^{-n+1}\kappa_q^2(1-q^n)(1-abcdq^{n-1})[2s]_q\right)I$$

where

$$G(s,a,b,c,d) = \sqrt{\prod_{i=1}^4(1-2q^{s_i}q^{-1/2}x(s-1/2)+q^{-1}q^{2s_i})}.$$

We now define

$$u(s,n) = D_n x_n(s) + D_n E_n + q^{-2s+1/2}\frac{(q^s-a)(q^s-b)(q^s-c)(q^s-d)}{[2s-1]_q}$$

where

$$D_n = -4q^{-n/2+1/2}(q-1)(1-abcdq^{n-1})$$
$$E_n = \frac{(-a-b-c-d+(abc+abd+acd+bcd)q^n)q^{n/2}}{2(1-abcdq^{2n})}.$$

Taking into account that $v(s,n) = u(s+1, n-1)$, we find

$$L^{+}(s,n) = u(s,n)I + \frac{2q^{3/2}}{[2s-1]_q} G(s,a,b,c,d)E^{-}$$
$$L^{-}(s,n) = v(s,n)I + \frac{2q^{3/2}}{[2s+1]_q} G(s+1,a,b,c,d)E^{+}$$

where $E^{-}f(s) = f(s-1)$ and $E^{+}f(s) = f(s+1)$. Thus,

$$L^{-}(s,n+1)L^{+}(s,n) = D_{2n}D_{2n+2}\gamma_{n+1}I + v(s,n+1)H(s,n)$$

and

$$L^{+}(s,n-1)L^{-}(s,n) = D_{2n-2}D_{2n}\gamma_n I + u(s,n-1)H(s,n)$$

which is the factorization formula for the Askey–Wilson functions.

To conclude this paper let us consider the special case of Askey–Wilson polynomials when $a = b = c = d = 0$: i.e., the continuous q-Hermite polynomials

$$H_n(x|q) = 2^{-n}\mathrm{e}^{\mathrm{i}n\theta}{}_2\phi_0\left(\begin{matrix} q^{-n}, 0 \\ — \end{matrix}\,\middle|\, q; q^n\mathrm{e}^{-2\mathrm{i}\theta}\right) \qquad x = \cos\theta.$$

These polynomials are closely related to the q-harmonic oscillator model introduced by Biedenharn [13] and Macfarlane [19], as was pointed out in [9], where the factorization for the continuous q-Hermite polynomials were first considered. If we substitute $a = b = c = d = 0$ in the above formulae, we obtain the factorization for the q-Hermite functions

$$\varphi_n(x) = \sqrt{\frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})(q^{n+1};q)_\infty}{2\pi\kappa_q(1-x^2)}}\, H_n(x|q).$$

In fact, since for continuous q-Hermite polynomials

$$\sigma(s) = -\kappa_q^2 q^{2s+1/2} \qquad \tau(s) = 4(q-1)x(s) \qquad \lambda_n = 4q^{-n+1}(1-q^n)$$

and the coefficients for the three-term recurrence relation are $\alpha_n = 1$, $\beta_n = 0$, $\gamma_n = (1-q^n)/4$, then we obtain

$$H(s,n) = \frac{2q^{3/2}}{[2s-1]_q}E^{-} + \frac{2q^{3/2}}{[2s+1]_q}E^{+}$$
$$+2\left(\frac{q^{-2s+1/2}}{[2s+1]_q} + \frac{q^{2s+1/2}}{[2s-1]_q} - q^{-n+1}\kappa_q^2(1-q^n)[2s]_q\right)I$$
$$L^{+}(s,n) = \left(-4q^{-n/2+1/2}(q-1)x(s+n/2) + \frac{q^{2s+1/2}}{[2s-1]_q}\right)I + \frac{2q^{3/2}}{[2s-1]_q}E^{-}$$
$$L^{-}(s,n) = \left(-4q^{-n/2+1}(q-1)x(s+n/2+1/2) + \frac{q^{2s+5/2}}{[2s+1]_q}\right)I + \frac{2q^{3/2}}{[2s+1]_q}E^{-}$$

and $h^{\pm}(n) = 4\kappa_q^2 q^{-2n+1}(1-q^n)$.

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Appendix

Here, for the sake of completeness, we prove proposition 4.1, by showing that $u(s+1,n)-v(s,n+1)=0$. To do that, we start by computing the difference

$$u(s+1,n)-v(s,n+1)=\frac{\lambda_n}{[n]_q}\frac{\tau_n(s+1)}{\tau_n'}-\frac{\Delta\sigma(s)}{\Delta x(s)}+\frac{\lambda_{n+1}}{[n+1]_q}\frac{\tau_{n+1}(s)}{\tau_{n+1}'}$$
$$-\lambda_{n+1}\Delta x\left(s-\frac{1}{2}\right)-\frac{\lambda_{2n+2}}{[2n+2]_q}(x(s)-\beta_{n+1})+\frac{\tau(s)\Delta x\left(s-\frac{1}{2}\right)}{\Delta x(s)}.$$

Now we use the expansion $\tau_n(s+1)=\tau_n' x_n(s+1)+\tau_n(0)$. Since

$$\frac{\Delta(x^2(s))}{\Delta x(s)}=\frac{x^2(s+1)-x^2(s)}{x(s+1)-x(s)}=x(s+1)+x(s)=C_1q^s(q+1)+C_2q^{-s}(q^{-1}+1)+2C_3$$
$$=(C_1q^{s+\frac{1}{2}}+C_2q^{-s-\frac{1}{2}})[2]_q+2C_3=[2]_qx_1(s)+(2-[2]_q)C_3$$
$$x(s)\Delta x(s-\tfrac{1}{2})=x(s)(C_1q^{s-\frac{1}{2}}(q-1)+C_2q^{-s+\frac{1}{2}}(q^{-1}-1))=x(s)(C_1q^s-C_2q^{-s})k_q$$
$$=(C_1^2q^{2s}-C_2^2q^{-2s})k_q+C_3(C_1q^s-C_2q^{-s})k_q$$

where $k_q=q^{\frac{1}{2}}-q^{-\frac{1}{2}}$,

$$\frac{\Delta}{\Delta x(s)}\left(x(s)\Delta x\left(s-\frac{1}{2}\right)\right)=\left(\frac{(C_1^2q^{2s+1}+C_2^2q^{-2s-1})[2]_q+C_3(C_1q^{s+\frac{1}{2}}+C_2q^{-s-\frac{1}{2}})}{C_1q^{s+\frac{1}{2}}-C_2q^{-s-\frac{1}{2}}}\right)k_q$$

and

$$\frac{\Delta}{\Delta x(s)}\left(\Delta x\left(s-\frac{1}{2}\right)\right)=\frac{\Delta}{\Delta x(s)}\Big((C_1q^s-C_2q^{-s})k_q\Big)=\frac{C_1q^{s+\frac{1}{2}}+C_2q^{-s-\frac{1}{2}}}{C_1q^{s+\frac{1}{2}}-C_2q^{-s-\frac{1}{2}}}k_q.$$

Then

$$\frac{\Delta\sigma(s)}{\Delta x(s)}=\frac{\Delta}{\Delta x(s)}\left(\tilde\sigma(s)-\frac{1}{2}\tilde\tau(s)\Delta x\left(s-\frac{1}{2}\right)\right)$$
$$=\frac{\Delta}{\Delta x(s)}\left(\frac{\tilde\sigma''}{2}x^2(s)+\tilde\sigma'(0)x(s)+\tilde\sigma(0)-\frac{1}{2}\left(\tau' x(s)+\tau(0)\right)\Delta x\left(s-\frac{1}{2}\right)\right)$$
$$=\frac{\tilde\sigma''}{2}\left([2]_qx_1(s)+(2-[2]_q)C_3\right)+\tilde\sigma'(0)-\frac{1}{2}\tau(0)\left(\frac{C_1q^{s+\frac{1}{2}}+C_2q^{-s-\frac{1}{2}}}{C_1q^{s+\frac{1}{2}}-C_2q^{-s-\frac{1}{2}}}\right)k_q$$
$$-\frac{1}{2}\tau'\left(\frac{[2]_q(C_1^2q^{2s+1}+C_2^2q^{-2s-1})+C_3(C_1q^{s+\frac{1}{2}}+C_2q^{-s-\frac{1}{2}})}{C_1q^{s+\frac{1}{2}}-C_2q^{-s-\frac{1}{2}}}\right)k_q.$$

This yields for $u(s+1,n)-v(s,n+1)$ the expression

$$=\left[\frac{\lambda_n}{[n]_q}x_n(s+1)+\frac{\lambda_n}{[n]_q}\frac{\tau_n(0)}{\tau_n'}\right]-\left[\frac{\tilde\sigma''}{2}[2]_qx_1(s)+\frac{C_3}{2}(2-[2]_q)\tilde\sigma''+\tilde\sigma'(0)\right.$$
$$-\frac{\tilde\tau'}{2}\left(\frac{[2]_q(C_1^2q^{2s+1}+C_2^2q^{-2s-1})}{C_1q^{s+\frac{1}{2}}-C_2q^{-s-\frac{1}{2}}}+\frac{C_3x_1(s)-C_3^2}{C_1q^{s+\frac{1}{2}}-C_2q^{-s-\frac{1}{2}}}\right)k_q$$
$$\left.-\frac{\tau(0)}{2}\left(\frac{x_1(s)-C_3}{C_1q^{s+\frac{1}{2}}-C_2q^{-s-\frac{1}{2}}}\right)k_q\right]+\frac{\lambda_{n+1}}{[n+1]_q}\frac{\tau_{n+1}(s)}{\tau_{n+1}'}-\lambda_{n+1}\Delta x\left(s-\frac{1}{2}\right)$$
$$-\frac{\lambda_{2n+2}}{[2n+2]_q}\left[C_1q^s+C_2q^{-s}+C_3-\frac{[n+1]_q\tau_n(0)}{\tau_n'}\right.$$
$$\left.+\frac{[n+2]_q\tau_{n+1}(0)}{\tau_{n+1}'}-C_3(1+[n+1]_q-[n+2]_q)\right]+\frac{\tau(s)\Delta x\left(s-\frac{1}{2}\right)}{\Delta x(s)}.$$

Next, we expand $\Delta x_n(s)$ and $\frac{\tilde{\sigma}''}{2}[2]x_1(s)$, make some straightforward calculations and use the identities

$$\frac{\lambda_n}{[n]_q}\frac{\tau_n(0)}{\tau_n'} + [n+1]_q\frac{\lambda_{2n+2}}{[2n+2]_q}\frac{\tau_n(0)}{\tau_n'} = \left(\frac{\lambda_n}{[n]_q} + [n+1]_q\frac{\lambda_{2n+2}}{[2n+2]_q}\right)\frac{\tau_n(0)}{\tau_n'} = -[n+2]_q\tau_n(0)$$

$$\frac{\lambda_{n+1}}{[n+1]_q}\frac{\tau_{n+1}(s)}{\tau_{n+1}'} - [n+2]_q\frac{\lambda_{2n+2}}{[2n+2]_q}\frac{\tau_{n+1}(0)}{\tau_{n+1}'} = [n+1]_q\tau_{n+1}(0) + \frac{\lambda_{n+1}}{[n+1]_q}x_{n+1}(s)$$

as well as

$$\begin{aligned}
&\frac{\lambda_n}{[n]_q}(C_1q^{s+1+\frac{n}{2}} + C_2q^{-s-1-\frac{n}{2}}) - \frac{\tilde{\sigma}''}{2}[2]_q(C_1q^{s+\frac{1}{2}} + C_2q^{-s-\frac{1}{2}}) - \frac{\lambda_{2n+2}}{[2n+2]_q}(C_1q^s + C_2q^{-s})\\
&\qquad -\lambda_{n+1}(C_1q^s - C_2q^{-s})k_q + \tfrac{1}{2}\tau'(C_1q^{s+\frac{1}{2}} + C_2q^{-s-\frac{1}{2}})(q+q^{-1})\\
&\qquad = \frac{C_1q^s\tau'}{2}(q^{n+\frac{1}{2}} + q^{\frac{1}{2}}) + \frac{C_1q^s\tilde{\sigma}''}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}(q^{n+\frac{1}{2}} - q^{\frac{1}{2}}) + \frac{C_2q^{-s}\tau'}{2}(q^{-n-\frac{1}{2}} + q^{-\frac{1}{2}})\\
&\qquad + \frac{C_2q^{-s}\tilde{\sigma}''}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}(-q^{-n-\frac{1}{2}} + q^{-\frac{1}{2}})\\
&\qquad = -\frac{\lambda_{n+1}}{[n+1]_q}(C_1q^{s+\frac{n+1}{2}} + C_2q^{-s-\frac{n+1}{2}})
\end{aligned}$$

and we find

$$\begin{aligned}
&= -\frac{\lambda_{n+1}}{[n+1]_q}\left(C_1q^{s+\frac{n+1}{2}} + C_2q^{-s-\frac{n+1}{2}}\right)\\
&\quad + C_3\frac{\lambda_n}{[n]_q} - [n+2]_q\tau_n(0) - C_3\tilde{\sigma}'' - \tilde{\sigma}'(0) + \frac{1}{2}\tau' C_3k_q\\
&\quad + \frac{1}{2}\tau(0)k_q + [n+1]_q\tau_{n+1}(0) + \frac{\lambda_{n+1}}{[n+1]_q}(C_1q^{s+\frac{n+1}{2}} + C_2q^{-s-\frac{n+1}{2}} + C_3)\\
&\quad + \frac{\lambda_{2n+2}}{[2n+2]_q}C_3([n+1]_q - [n+2]_q).
\end{aligned}$$

Finally, we substitute the expression for $\tau_n(0)$ and use the identities

$$-[n+2]_q[n]_q - 1 + [n+1]_q[n+1]_q = 0$$
$$-[n+2]_q(q^{n/2} + q^{-n/2}) + k_q + [n+1]_q(q^{(n+1)/2} + q^{(n+1)/2}) = 0$$

and the result follows.

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